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Time-Dependent Flow of Quasi-Linear Viscoelastic Fluids

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Synopsis

Transient flow behavior of an incompressible quasi-linear viscoelastic fluid under suddenly applied constant pressure as well as under a periodic pressure gradient was investigated using a three-parameter relaxation function. In the light of these solutions, the roll of viscoelastic relaxation in the overshooting of volumetric flow rate and the effect of viscoelastic parameters on the mean square velocity profile are discussed.

INTRODUCTION

The transient flow behavior of linear viscoelastic fluids under small stress and deformation rate was studied by several investigators.[1–5] The rheologic equations used in all of these studies assume some specific forms of the material functions. For example, using one of the simplest forms of the Oldroyd equations[7] with three rheologic constants, Chong and Vezzi[5] showed that the stress equation of motion for the flow under a constant pressure gradient can be transformed to an equation similar to the forced vibration of an elastic material. Based on this equation they discussed several possibilities in transient flow behavior.

In the present work we use an integral rheologic equation for linear viscoelasticity proposed by Fredrickson[10] and Lodge.[14]

$$\sigma^{ij}(x, t) = 2 \int_{-\infty}^{t} \psi(t - t') \frac{\partial x^i}{\partial X^m} \frac{\partial x^j}{\partial X^n} \dot{e}^{mn}(X, t')dt' \qquad (1)$$

where $\sigma^{ij}(x, t)$ is the stress tensor; $\psi(t - t')$ is the relaxation function; X^i are the material coordinates, that is, the material point which has coordinates x^i at time t has coordinates X^i at time $t'(\leq t)$; and $\dot{e}^{mn}(X, t')$ is the rate of strain tensor at x and t'.

Fredrickson[10] shows that the three-constant Oldroyd fluid under a small stress and deformation rate is a special case of eq. (1) with relaxation function given by

$$\psi(t - t') = \frac{\eta}{\lambda_1} \left[\left(1 - \frac{\lambda_2}{\lambda_1}\right) e^{-(t-t')/\lambda_1} + \lambda_2 \delta(t - t') \right] \qquad (2)$$

where $\delta(t)$ denotes the Dirac delta function and λ_1 and λ_2 are the relaxation and retardation times, respectively.

According to Fredrickson,[10] the material described by eq. (1) would essentially behave like Newtonian fluids in steady laminar shear flow if the relaxation function satisfies the following condition:

$$\eta = \int_0^\infty \psi(t)dt \tag{3}$$

$$\alpha = \int_0^\infty t\psi(t)dt \tag{4}$$

where eq. (3) describes the steady state Newtonian viscosity and eq. (4) is the normal stress coefficient. If these integrals do not exist, the material is an elasticoviscous solid and cannot exhibit steady flow without fracture.

Though eq. (1) is motivated by classical linear viscoelasticity for small deformation, it is important to note that the restriction of small deformation is not inherent in the equation and the principle of invariance[9] is satisfied. Therefore, it may be called quasi-linear viscoelasticity.

In this work, we consider two transient flow problems of viscoelastic materials whose stress–deformation relation follows eq. (1) with relatively simple relaxation functions. The problems to be studied are the startup of an incompressible fluid in a cylindrical tube under a constant pressure gradient as well as under a periodic pressure gradient.

FLOW UNDER SUDDENLY IMPOSED CONSTANT PRESSURE

Let us consider the transient flow of this fluid in a long cylindrical tube of constant cross section. The fluid is at rest in the tube for $t \leq 0$; and at $t > 0$, a constant pressure is suddenly applied to one end of the tube and maintained. If we assume the fluid is incompressible, the physical components of the velocity in cylindrical coordinates are[9]

$$U_r = U_\theta = 0 \quad \text{for all } t$$

$$U_Z = \begin{cases} 0 & t \leq 0 \\ U(r, t) & t > 0. \end{cases}$$

The material and current coordinates are

$$r = x^1 = X^1$$

$$\theta = x^2 = X^2$$

$$Z = x^3 = X^3 + \int_{t'}^{t} U(X', \tau)d\tau.$$

The rate of strain tensor is given by

$$\|\dot{e}^{mn}(X, t')\| = \frac{1}{2}\frac{\partial U(r, t')}{\partial r}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

From eq. (1) we obtain shearing stress and normal stress components, respectively:

$$\sigma_{rZ} = \int_0^t \psi(t - t') \frac{\partial U(r, t')}{\partial r} dt' \tag{5}$$

$$\sigma_{ZZ} = 2 \int_0^t \int_{t'}^t \psi(t - t') \frac{\partial U(r, t')}{\partial r} \frac{\partial U(r, t')}{\partial r} d\tau dt'. \tag{6}$$

The equation of motion in cylindrical coordinates is

$$\rho \frac{\partial U}{\partial t} = - \frac{\partial P}{\partial Z} + \frac{1}{r} \left[r \int_0^t \psi(t - t') \frac{\partial U(r, t')}{\partial r} \right] dt'. \tag{7}$$

Since the pressure gradient in eq. (7) is maintained constant for $t > 0$, it is independent of time.

We first convert eq. (7) to an integro-differential equation by multiplying $rJ_0(\zeta_i r)$ and integrating the resulting equation with respect to r from 0 to a, where a is the tube radius:

$$\rho \frac{\partial}{\partial t} \int_0^a r\, UJ_0(\zeta_i r) dr = \Phi \int_0^a rJ_0(\zeta_i r) dr + \int_0^a J_0(\zeta_i r) \frac{\partial}{\partial r} \left[r \int_0^t \psi(t - t') \frac{\partial U}{\partial r} dt' \right] dr \tag{8}$$

where $\Phi = -\dfrac{\partial P}{\partial Z}$.

Let $U_H(\zeta_i, t) = \int_0^a rUJ_0(\zeta_i r)dr;$ then it is easy to show that eq. (8) reduces to

$$\rho \frac{dU_H}{dt} = \Phi \frac{a}{\zeta_i} J_1(\zeta_i a) + \zeta_i \int_0^a J_1(\zeta_i r) r \int_0^t (t - t') \frac{\partial U}{\partial r} dt' dr \tag{9}$$

where ζ_i satisfies $J_0(\zeta_i a) = 0$.

We can rearrange the order of integration of eq. (9) to give

$$\rho \frac{dU_H}{dt} = \Phi \frac{a}{\zeta_i} J_1(\zeta_i a) + \zeta_i \int_0^t \psi(t - t') \int_0^a r \frac{\partial U(r, t')}{\partial r} J_1(\zeta_i r) dr dt'. \tag{10}$$

Equation (10) can be further reduced to

$$\frac{dU_H}{dt} + \frac{\zeta_i^2}{\rho} \int_0^t \psi(t - t') U_H(\zeta_i, t') dt' = \Phi \frac{aJ_1(\zeta_i a)}{\rho \zeta_i}. \tag{11}$$

Equation (11) states that a solution for the velocity distribution at present time t requires knowledge of the velocity distribution at a time t' prior to t. This difficulty can be avoided, however, if we take the Laplace transform of eq. (11) and note that the second term on the left side of the equation is a convolution integral. Thus we obtain

$$S\bar{U}_H(\zeta_i, S) + \frac{\zeta_i^2}{\rho}\bar{\psi}(S)\bar{U}_H(\zeta_i, S) = \Phi\frac{aJ_1(\zeta_i, a)}{S\zeta_i\rho} \tag{12}$$

where S is the Laplace transform variable, and the initial condition $U_H(\zeta_i, 0) = 0$ is used. From eq. (12) it follows that

$$S\bar{U}_H(\zeta_i, S) = \frac{\Phi a J_1(\zeta_i a)/\rho\zeta_i}{S + (\nu\zeta_i^2/\eta)\bar{\psi}(S)} \tag{13}$$

where ν is the kinematic viscosity. It is necessary to obtain $U_H(\zeta_i, t)$ by inverse transform of eq. (13). Before we do so, it may be of interest to study the validity of eq. (13) in the limiting case where $t \to \infty$, the steady state. For this purpose we apply one of the important properties of the Laplace transforms which states that

$$\lim_{s\to 0} S\overline{f(S)} = \lim_{t\to\infty} f(t). \tag{14}$$

Thus we see eq. (13) can be reduced to

$$\lim_{s\to 0} S\bar{U}_H(\zeta_i, S) = \lim_{t\to\infty} U_H(\zeta_i, t) = \frac{\Phi a J_1(\zeta_i a)}{\zeta_i^3\eta} \tag{15}$$

where

$$\lim_{s\to 0}\overline{\psi(S)} = \int_0^\infty \psi(t)dt = \eta. \tag{16}$$

Now we apply the Hankel inversion theorem[8] to eq. (15) to obtain the velocity distribution at steady states:

$$U(r, \infty) = \frac{2\Phi}{\eta a}\sum_i \frac{J_0(\zeta_i r)}{\zeta_i^3 J_1(\zeta_i a)} = \frac{\Phi a^2}{4\eta}\left[1 - \left(\frac{r}{a}\right)^2\right] \tag{17}$$

where the summation extends to all positive roots of ζ_i.

Equation (17) is indeed the familiar parabolic velocity distribution for Newtonian fluids. Therefore, the validity of eq. (13) in the limiting case is verified.

Returning to eq. (13) we obtain an expression for $U_H(\zeta_i, t)$ by inverse Laplace transform. Symbolically this is denoted as

$$U_H(\zeta_i, t) = \Phi\frac{aJ_1(\zeta_i a)}{\rho\zeta_i} L^{-1}\left\{\frac{1/S}{S + (\nu\zeta_i^2/\eta)\bar{\psi}(S)}\right\}. \tag{18}$$

The velocity distribution is now obtained by applying the Hankel inversion theorem to eq. (18). The reader should realize that we are using a double transform technique to solve for the velocity. Of course the function $U(r, t)$ must satisfy Dirichlet's condition[8] in the intervals $(0, a)$ for the inversion theorem to be valid:

$$U(r, t) = \Phi\frac{2\nu}{\eta a}\sum_i \frac{1}{\zeta_i}\frac{J_0(\zeta_i r)}{J_1(\zeta_i a)} L^{-1}\left\{\frac{1/S}{S + (\nu\zeta_i^2/\eta)\overline{\psi(S)}}\right\}. \tag{19}$$

The actual inverse transform of eq. (19) may be otained if we know the Laplace transform of the relaxation function, $\overline{\psi(S)}$. Equation (19) shows clearly how the stress relaxation function affects the velocity distribution during the transient flow.

As a simple example, let us assume the relaxation function is given by eq. (2). The Laplace transform of the relaxation function is

$$\overline{\psi(S)} = \eta\left(\frac{1 + \lambda_2 S}{1 + \lambda_1 S}\right). \tag{20}$$

Substituting eq. (20) into (19), we obtain

$$U(r, t) = \frac{2\nu\Phi}{\eta}\sum_i \frac{1}{(a\zeta_i)}\frac{J_0(\zeta_i r)}{J_1(\zeta_i a)} L^{-1}\left\{\frac{1}{S}\cdot\frac{1 + \lambda_1 S}{\lambda_1 S^2 + (1 + \nu\zeta_i^2)S + \nu\zeta_i^2}\right\}. \tag{21}$$

The inverse transform of eq. (21) can be obtained readily by combination of the Heaviside partial fractions theorem and the convolution integral. There are three possible solutions for the equation depending on the values of λ_1 and λ_2. Here we will examine two of them:

$$\frac{1}{\lambda_1} + \nu^2\zeta_i^4\lambda_2\left(\frac{\lambda_2}{\lambda_1}\right) \gtrless 2\nu\zeta_i^2\left(2 - \frac{\lambda_2}{\lambda_1}\right). \tag{22a, b}$$

The velocity distribution for the condition (22a) becomes

$$\begin{aligned} U(r, t) = \frac{2\Phi a^2}{\eta}\sum_i \frac{1}{(\zeta_i a)^3}\frac{J_0(\zeta_i r)}{J_1(\zeta_i a)}\Bigg\{\frac{2 - \alpha_i}{\sqrt{\alpha_i^2 - \beta_i}}\Bigg[N\lambda_1 - e^{(-\alpha_i t/2\lambda_1)} \\ \times\left(\frac{\alpha_i}{2}\times \sinh Nt + \lambda_1 N\cosh Nt\right)\Bigg] \\ + \left(\frac{\alpha_i}{2}\right) - e^{(-\alpha_i t/2\lambda_1)}\left(\frac{\alpha_i}{2}\cosh Nt + \lambda_1 N \sinh Nt\right)\Bigg\} \end{aligned} \tag{23}$$

where $\alpha_i = 1 + \nu\zeta_i^2\lambda_2$, $\beta_i = 4\lambda_1\nu\zeta_i^2$, and $N = \dfrac{1}{2\lambda_1}(\alpha_i^2 - \beta_i)^{1/2}$.

This condition implies that $\alpha_i^2 - \beta_i > 0$. If the ratio of λ_2/λ_1 approaches zero, then we have $\beta_i - \alpha_i^2 > 0$, and the velocity distribution for the condition (22b) becomes

$$\begin{aligned} U(r, t) = \frac{2\Phi a^2}{\eta}\sum_i \frac{1}{(a\zeta_i)^3}\frac{J_0(\zeta_i r)}{J_1(\zeta_i a)}\Bigg\{\frac{2 - \alpha_i}{\sqrt{\alpha_i^2 - \beta_i}}\Bigg[|N|\lambda_1 - e^{-\alpha_i t/2\lambda_1} \\ \times\left(\frac{\alpha_i}{2\lambda_1}\sin |N|t + \lambda_1 N\cos |N|t)\right)\Bigg] \\ + \frac{\alpha_i}{2} - e^{(-\alpha_i t\lambda/2\lambda_1)}\frac{\alpha_i}{2}\left(\cos |N|t - \lambda_1|N|\sin |N|t\right)\Bigg\}. \end{aligned} \tag{24}$$

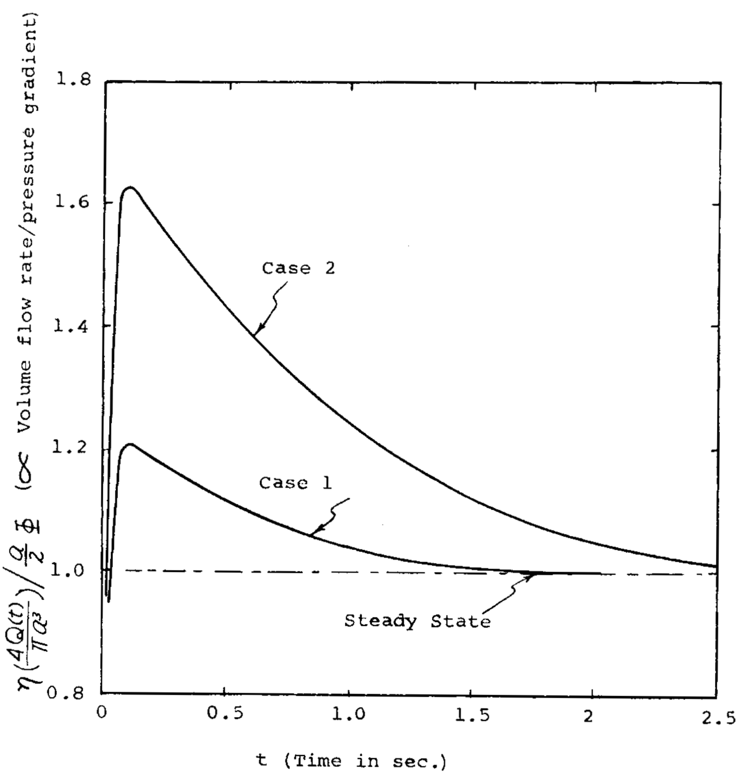

Fig. 1. Instantaneous volumetric flow rate (dimensionless) as a function of time. Case 1: π_1, 0.75 sec; π_2, 0.60 sec; ν_1 10.4 cm^2/sec; a, 1.0 cm. Case 2: λ_1, 1.6 sec; λ_2, 0.04 sec; γ_1 15.0 cm^2/sec; α, 1.0 cm.

In the limiting case where $\lambda_2 = 0$, eq. (24) reduces to a Maxwell fluid.

The volumetric flow rates, $Q(t)$, for conditions (22a,b) are obtained from eqs. (23) and (24). They are, respectively,

$$\frac{Q(t)}{\pi a^3} = \frac{4\Phi a}{\eta} \sum_i \frac{1}{(a\zeta_i)^4} \left\{ \frac{2 - \alpha_i}{\sqrt{\alpha_i^2 - \beta_i}} \left[\lambda_1 N - e^{(-\alpha_i t/2\lambda_1)} \times \left(\frac{\alpha_i}{2} \sinh Nt + \lambda_1 N \cosh Nt \right) \right] + \frac{\alpha_i}{2} - e^{(-\alpha_i t/2\lambda_1)} \left(\frac{\alpha_i}{2} \cosh Nt + \lambda_1 N \sinh Nt \right) \right\} \quad (25a)$$

$$\frac{Q(t)}{\pi a^3} = \frac{4\Phi a}{\eta} \sum_i \frac{1}{(a\zeta_i)^4} \left\{ \frac{2 - \alpha_i}{\sqrt{\alpha_i^2 \beta_i}} \left[\lambda_1 N - e^{(-\alpha_i t/2\lambda_1)} \times \left(\frac{\alpha_i}{2\lambda} \sin |N| t + \lambda_1 N \cos |N| t \right) \right] + \left(\frac{\alpha_i}{2} \right) - e^{(-\alpha_i t/2t\lambda_1)} \left(\frac{\alpha_i}{2} \cos |N| t - \lambda_1 |N| \sin |N| t \right) \right\}. \quad (25b)$$

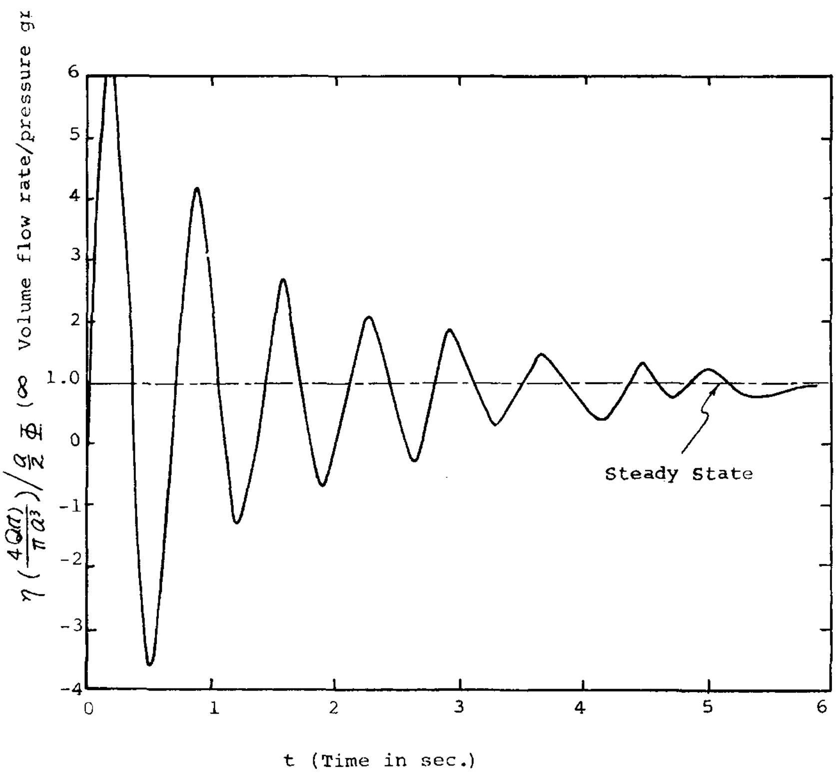

Fig. 2. Instantaneous volumetric flow rate as a function of time: λ_1, 0.75 sec; λ_2, 0 sec; ν, 10.4 cm^2/sec; a, 1.0 cm.

Equations (25a, b) are computed using several different values of rheologic parameters and the results are shown in Figures 1 and 2. Figure 1 shows overshoots in volumetric flow rate and qualitatively this is in agreement with experimental result reported by Meissner.[13]

As the ratio of retardation to relaxation time approaches zero, the volumetric flow rate exhibits damped oscillatory motion. This is shown in Figure 2.

The normal stress component can now be computed by substituting eq. (23) or (24) into (6), but we will omit the resulting final equation.

FLOW UNDER VARYING PRESSURE

Let us consider another problem in which the fluid is at rest in a long cylindrical tube for $t \leq 0$, and for $t > 0$ it is pushed out by a frictionless plunger at a constant speed. An example of this type of flow is encountered in an Instron capillary rheometer. We are interested in a relationship between pressure variation and velocity development. Under this condition eq. (8) becomes

$$\frac{dU_H}{dt} + \frac{\zeta_i^2}{\rho}\int_0^t \psi(t - t')U_H(\zeta_i, t')dt' = \Phi(t)\,\frac{aJ_1(\zeta_i a)}{\rho\zeta_i} \tag{26}$$

where $\Phi(t)$ denotes the time dependent pressure gradient. We take the Laplace transform of eq. (26) and obtain

$$S\bar{U}_H(\zeta_i, S) + \frac{\zeta_i^2}{\rho}\,\overline{\psi(S)U_H}(\zeta_i, S) = \overline{\Phi(S)}\,\frac{aJ_1(\zeta_i a)}{\rho\zeta_i}. \tag{27}$$

Solving for $\bar{U}_H(\zeta_i, S)$, we obtain

$$\bar{U}_H(\zeta_i, S) = \frac{\Phi(\bar{S})aJ_1(\zeta_i a)/\rho\zeta_i}{S + (\nu\zeta_i^2/\eta)/\psi\bar{S}}. \tag{28}$$

The inverse transform of eq. (28) is denoted as

$$U_H(\zeta_i, t) = \frac{aJ_1(\zeta_i a)}{\rho\zeta_i}\,L^{-1}\left\{\frac{\overline{\Phi(S)}}{S + (\nu\zeta_i^2/\eta)\psi(\bar{S})}\right\}. \tag{29}$$

The velocity distribution can now be obtained by applying the Hankel inversion theorem to eq. (29). Thus, we have

$$U(r, t) = \frac{2\nu}{\eta}\sum_i \frac{1}{a\zeta_i}\frac{J_0(\zeta_i r)}{J_1(\zeta_i a)}\,L^{-1}\left\{\frac{\overline{\Phi(S)}}{S + (\nu\zeta_i^2/\eta)\overline{\psi(S)}}\right\} \tag{30}$$

where the summation extends to all positive roots ζ_i. Equation (30) shows how the pressure variation and the relaxation function both affect the velocity distribution during the transient flow.

As a simple example, we assume again that the relaxation function is given by eq. (20). Substitution of eq. (20) into (30) yields

$$U(r, t) = \frac{2\nu}{\eta}\sum_i \frac{1}{(a\zeta_i)}\frac{J_0(\zeta_i, r)}{J_1(\zeta_i a)}\,L^{-1}\left\{\overline{\Phi(S)}\cdot\frac{1 + \lambda_1 S}{\lambda_1 S^2 + \alpha_i S + \zeta_i^2\nu}\right\}. \tag{31}$$

Since the inverse transform of the relaxation part (the right-hand side term in the bracket) of the above equation is known in the previous example, we can express eq. (31) in terms of the convolution integral. Here we consider a solution of eq. (31) for the condition (24a) where $\alpha_i^2 - \beta_i > 0$. Equation (31) becomes:

$$\begin{aligned} U(r, t) = \frac{2\nu}{\eta}\sum_i \frac{1}{(a\zeta_i)}\frac{J_0(\zeta_i r)}{J_1(\zeta_i a)}&\left\{\frac{2 - \alpha_i}{\sqrt{\alpha_i^2 - \beta_i}}\int_0^t \Phi(t')\,e^{[-\alpha_i/2\lambda_1(t - t')]}\right. \\ &\times \sinh\frac{(\alpha_i^2 - \beta_i)^{1/2}(t - t')}{2\lambda_1}\,dt' + \int_0^t \Phi(t')\,e^{[-\alpha_i(t-t')/2\lambda_1]} \\ &\left.\times \cosh\frac{(\alpha_i^2 - \beta_i)^{1/2}(t - t')}{2\lambda_1}\,dt'\right\} \end{aligned} \tag{32}$$

where $\Phi(t')$ is the pressure gradient as a function of variable time, t'.

Equation (32) gives a relationship between pressure and velocity distribution. It should be noted that unless the variation of pressure with respect to time is known, it is not possible to calculate the velocity profile using eq. (32). It appears, therefore, that the velocity profile and the volumetric flow rate for this case cannot be computed using the above approach. This problem, however, has been solved elsewhere[1,5] by using a differential form of the Oldroyd equation. It may be noted, however, that the exact solutions for the velocity distribution and volumetric flow rate can be computed from eq. (32) if the flow is under a periodic pressure. To show this, we now consider the effect of viscoelastic properties on oscillating flow.

OSCILLATING FLOW

Let us assume that the periodic pressure gradient is

$$\Phi(t') = K \cos \omega t' \tag{33}$$

where K is a constant and ω is the frequency. The velocity distribution under this condition becomes

$$U(r, t) = \frac{\nu K}{\eta} \sum_i \frac{1}{(a\zeta_i)} \frac{J_0(\zeta_i r)}{J_1(\zeta_i a)} \left\{ \frac{\left((1 + \frac{2 - \alpha_i}{\sqrt{\alpha_i^2 - \beta_i}}\right)(A_i \cos \omega t + \omega \sin \omega t}{(\omega^2 + A_i^2)} \right.$$

$$\left. - A_i e^{-A_i t}) + \frac{\left(1 - \frac{2 - \alpha_i}{\sqrt{\alpha_i^2 - \beta_i}}\right)(B_i \cos \omega t + \omega \sin \omega t - B_i e^{-B_i t})}{(\omega^2 + A_i^2)} \right\} \tag{34}$$

where $A_i = (1/2\lambda_1)(\alpha_i - \sqrt{\alpha_i^2 - \beta_i})$ and $B_i = (1/2\lambda_1)(\alpha_i + \sqrt{\alpha_i^2 - \beta_i})$. When the system reaches steady state long time after the flow started, eq. (34) reduces to

$$U(r, t) = \frac{\nu K}{\omega \eta} \sum_i \frac{1}{(a\zeta_i)} \frac{J(_0\zeta_i r)}{J_1(\zeta_i a)} \{\omega \cos \omega t \cdot (A_i K_1 + B_i K_2)$$

$$+ \sin \omega t \cdot (K_1 \omega + K_2 \omega)\} \tag{35}$$

where $K_1 = \left(1 + \frac{2 - \alpha_i}{\sqrt{\alpha_i^2 - \beta_i}}\right) \Big/ [(\omega^2 + A_i^2)]$ and

$$K_2 = \left(1 - \frac{2 - \alpha_i}{\sqrt{\alpha_i^3 - \beta_i}}\right) \Big/ [(\omega^2 + \beta_i^2)]. \tag{35}$$

In the limiting case where the frequency is very small, eq. (35) reduces to

$$U(r, t) = \frac{2K}{a\eta} \cos \omega t \sum_i \frac{1}{\zeta_i^3} \frac{J_0(\zeta_i r)}{J_1(\zeta_i a)} = \frac{Ka^2}{4\eta} \left[1 - \left(\frac{r}{a}\right)^2\right] \cos \omega t. \tag{36}$$

The velocity distribution becomes parabolic, as expected, and the flow is in phase with the pressure.

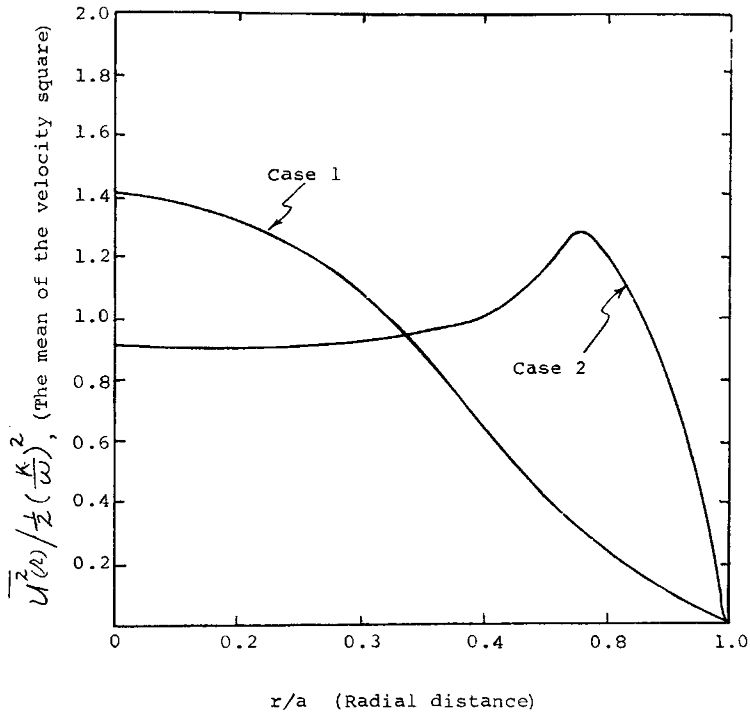

Fig. 3. Mean of velocity square profile for a Newtonian fluid. Case 1: ν_1 1.52 cm²/sec; a, 1.0 cm; ω, 15.5 cycles/sec. Case 2: ν, 1.52 cm²/sec; a, 1.0 cm; ω, 15.5 cycles/sec.

The volumetric flow rate can be obtained readily from eq. (36):

$$\begin{aligned} Q(t) &= 2\pi \int_0^a rU(r, t)dr \\ &= \frac{2\pi K \nu a^2}{\eta} \sum_i \frac{1}{(a\zeta_i)} \sqrt{(K_i A_i + K_2 B_i)^2 + \omega^2(K_1 + K_2)^2} \\ &\qquad \times \sin \omega(t + \delta_i). \end{aligned} \quad (37)$$

Equation (37) shows that each eigenvalue has its corresponding amplitude and phase angle. This fact suggests that the evaluation of viscoelastic parameters based on the experimental measurements of amplitude and phase angle is not practical unless eq. (37) converges very rapidly.

It is of interest to compare eq. (36) with the velocity distribution of a Newtonian fluid which is derived in this work using the same method:

$$U(r, t) = \frac{2\nu K}{\omega\eta} \sum_i \frac{J_0(\zeta_i, r)}{(a\zeta_i)J_1(\zeta_i, a)} \frac{\left(\frac{\nu\zeta_i^2}{\omega} \cos \omega t + \sin \omega t\right)}{[1 + (\nu\zeta_i^2/\omega)^2]}. \quad (38)$$

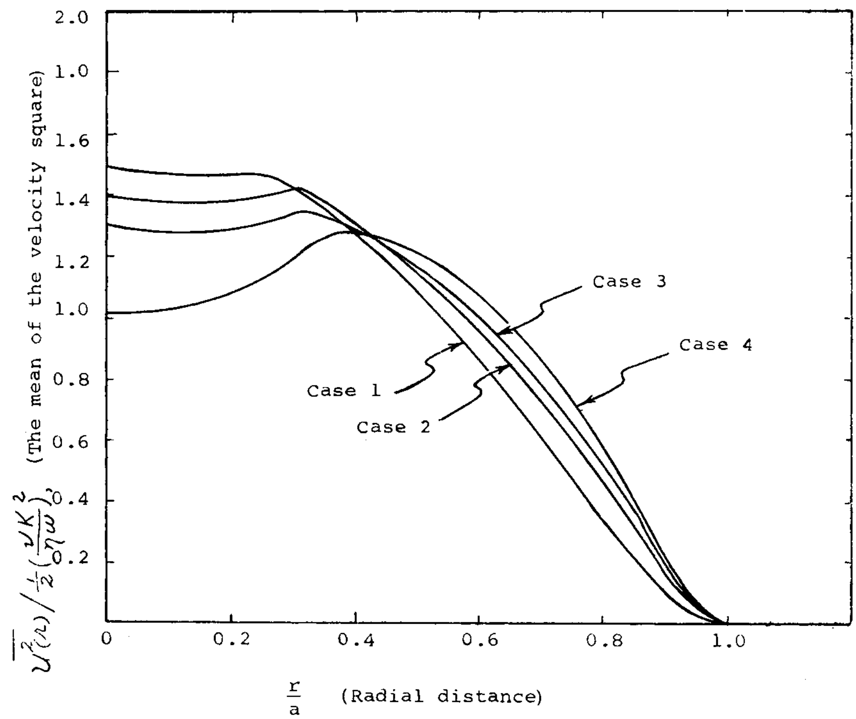

Fig. 4. Mean of velocity square profiles for visocelastic fluids. Case 1: ν, 1.52; λ_1, 5.03 × 10^{-2}; λ_2, 3.35 × 10^{-2}; ω, 360; a, 0.2. Case 2: ν_1 1.52; λ_1, 0.172; λ_2, 7.763 × 10^{-2}; ω_1 360; a_1 0.2. Case 3: ν_1 1.52; λ_1, 5.03 × 10^{-2}; λ_2, 3.35 × 10^{-2}; ω_1 620; a, 0.2. Case 4: ν_1 1.52; λ_1, 0.172; λ_2, 7.76 × 10^{-2}; ω_1 620; a, 0.2. Units same as in Figs. 1–3.

We are interested in the mean with respect to time of the velocity square which is defined as

$$\overline{U^2(r)} = \frac{1}{t}\int_0^t U^2(r, t)dt. \tag{39}$$

From eqs. (36) and (39), we find that the exact expression for the mean of the velocity square is

$$\frac{\overline{U^2(r)}}{{}^1/_2(\nu K/\omega\eta)^2} = \left(\sum_i \frac{\omega}{(a\zeta_i)} \frac{J_0(\zeta_i r)}{J_1(\zeta_i a)} [A_i K_1 + \beta_i K_2]\right)^2 + \left(\sum_i \frac{\omega^2 J_0(\zeta_i r)}{(a\zeta_i)J_1(\zeta_i a)} [K_1 + K_2]\right)^2 \cdot \tag{40}$$

For Newtonian fluids at a high frequency it is known[11] that the mean of the velocity square shows a maximum near the wall at $(a - r)\sqrt{\omega/2\nu} = 2.28$. This is shown in Figure 3. The mean of the dimensionless velocity square, $\overline{U^2(r)}/[{}^1/_2(\nu K/\omega\eta)^2]$, for a viscoelastic fluid is computed as a function of radial distance and the result is shown in Figure 4. The figure shows that at a given frequency the mean of the velocity square is significantly different from Newtonian fluids. For Newtonian fluids, the velocity profile across the tube radius flattens out and the maximum occurs near the wall, while for viscoelastic fluids this maximum occurs near the tube center. As the frequency is increased, this maximum point moves toward the wall. This seems to indicate that viscoelastic properties have a stabilizing effect on velocity profile. It is suggested that this mechanism might cause the delay in transition from laminar to turbulent flow.

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